(6–8) that $\bar{V} = 1$, $T/mg = \frac{1}{2} \sin \gamma$, and $e = \sin \gamma$. Therefore, it follows that the velocity decreases along the spiral path but is always equal to the value of local circular orbital speed whereas the value of T/mg must remain fixed along with the eccentricity of the osculating ellipse. From the results presented in Table 3 for the effects of continuous low thrust (with T/mg fixed) applied to a vehicle in circular orbit, it is found that the maximum value of flight-path angle is almost exactly twice the value for the corresponding logarithmic spiral so that the average value of flight-path angle is about the same as the fixed spiral angle. Since the average value of $\bar{V}$ in each case presented in Table 3 is also the same as the fixed value for the logarithmic spiral, it is apparent that the flight path in each case of continuous low thrust may be considered to oscillate around the corresponding logarithmic spiral trajectory.

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Optimum Rendezvous Transfer Between Coplanar Heliocentric Elliptic Orbits Using Solar Sail

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Introduction

IT is well recognized that solar-sail vehicles, using light pressure for propulsion, offer an attractive mode of travel for interplanetary space probes. The parameter that best indicates sail spacecraft's performance capability is the characteristic acceleration α, which the spacecraft would experience at a unit distance (1 AU) from the sun when the sail is oriented normal to sunlight. Feasible values for α seem to be 0.5–3.0 mm/s^2. For an idealized, perfectly reflecting flat sail whose area and spacecraft mass remain invariant with time, the local acceleration of the spacecraft would vary by the square of the cosine of the sail orientation angle (θ angle between the sail's outward drawn normal and the radial direction) and inversely as the square of the heliocentric distance, r. It is interesting to evaluate the time history of θ that would fulfill a certain mission in a prescribed optimal manner. The present study is concerned with the time-optimal rendezvous transfer trajectories. The analysis, however, is restricted to a two-body inverse square law of force field model in two dimensions. Employing calculus of variations in the form of "maximum principle," Zhukov and Lebedev[1] presented a minimum time strategy for transfer between two coplanar, circular, heliocentric orbits. The present attempt is an expansion to include elliptic orbits for the terminals. The formulation is attempted through a set of autonomous variables L, Φ, and Ψ [see Eq. (3)] related to angular momentum h and the Cartesian components of the eccentricity vector. As in Zhukov and Lebedev, the optimal strategy is pursued by conversion to a two-point boundary-value problem for a system of seven ordinary differential equations. Its solution is attempted with the controlled random search (CRS) optimization technique. It can be noted that Sauer[2] has presented a formulation in terms of the position and velocity vectors in three dimensions. However, the present formulation brings out explicitly the effect of the eccentricities of the terminal orbits on the time-optimal transfer.

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Model for Space Vehicle Motion

Following the approach of vector techniques,[3] we consider the equations of motion of a spacecraft with an idealized, perfectly reflecting flat sail, in terms of the radial and transversal components Q and R, respectively, of the perturbing acceleration:

$$h' = h^5 r/P^3 \tag{1a}$$

$$\delta' = \left[PQ \sin\phi + R(1+P)\cos\phi + R\delta\right] h^4/P^3 \tag{1b}$$

$$\epsilon' = \left[-PQ\cos\phi + R(1+P)\sin\phi + R\epsilon\right] h^4/P^3 \tag{1c}$$

where

$$P = 1 + \delta\cos\phi + \epsilon\sin\phi, \qquad r = h^2/P$$

$$Q = \alpha\cos^3\theta/r^2, \qquad R = \alpha\cos^2\theta\sin\theta/r^2$$

and δ, ϵ are the components $e\cos\omega$, $e\sin\omega$ of the eccentricity (e) vector, ω being the argument of perihelian. The prime indicates differentiation with respect to the angular position variable ϕ of the vehicle; this is related to the time t by

$$t' = h^3/P^2 \tag{2}$$

The units employed for the distance and velocity are astronomical units (1496×10^5 km) and Earth-mean-orbital speed (29.78 km/s). Although this description provides a good stable foundation for orbit development, the equations are considerably simplified[4] when we transform the parameters h, δ, and ϵ to the variables defined by

$$L = \ln(h^2) \tag{3a}$$

$$\Phi = \delta\cos\phi + \epsilon\sin\phi = e\cos\nu \tag{3b}$$

$$\Psi = \delta\sin\phi - \epsilon\cos\phi = e\sin\nu \tag{3c}$$

where ν is the true anomaly. The transformed equations of motion are

$$L' = 2\alpha\cos^2\theta\sin\theta/P \tag{4a}$$

$$\Phi' = 2\alpha\cos^2\theta\sin\theta - \Psi \tag{4b}$$

$$\Psi' = \Phi + \alpha\cos^2\theta[\cos\theta + \sin\theta\Psi/P] \tag{4c}$$

$$t' = \exp(3L/2)/P^2 \tag{4d}$$

Problem Formulation

Initially (at $t=0$), the spacecraft is in the same heliocentric elliptic orbit as that of the Earth (departure planet), with the initial conditions

$$\phi = \phi_1, \qquad L = L_1 = \ln(h_1^2) \tag{5a}$$

$$\Phi = \Phi_1 = e_1 \cos \nu_1, \qquad \Psi = \Psi_1 = e_1 \sin \nu_1 \tag{5b}$$

The vehicle is to be transferred to the target planet's orbit with terminal conditions

$$\phi = \phi_2, \qquad L = L_2 = \ln(h_2^2) \tag{6a}$$

$$\Phi = \Phi_2 = e_2 \cos(\phi_2 - \omega_2), \qquad \Psi = \Psi_2 = e_2 \sin(\phi_2 - \omega_2) \tag{6b}$$

where ϕ_2 is free, and the subscripts 1 and 2 correspond to the departure and arrival planets. The problem now is to find a suitable profile $\theta(\phi)$ for transfer from a point where Eq. (5) holds to a point where Eq. (6) holds in minimum time (T_F).

Solution Methodology

Solution is attempted by conversion to a two-point boundary-value problem utilizing the procedures based on Pontryagin's principle.[5] Introducing the adjoint variables p_1, p_2, p_3, and p_4 corresponding to L, Φ, Ψ, and t, respectively, the Hamiltonian H for the problem is set as

$$H = p_1 L' + p_2 \Phi' + p_3 \Psi' + p_4 t' \tag{7}$$

The adjoint system is

$$p_1' = -\frac{\partial H}{\partial L} = -1.5 p_4 \exp(3L/2)/P^2 \tag{8a}$$

$$p_2' = -\frac{\partial H}{\partial \Phi} = -p_3 + 2p_4 \exp(3L/2)/P^3$$

$$+ \alpha \cos^2\theta \sin\theta (2p_1 + p_3 \Psi)/P^2 \tag{8b}$$

$$p_3' = -\frac{\partial H}{\partial \Psi} = p_2 - p_3 \alpha \cos^2\theta \sin\theta / P \tag{8c}$$

$$p_4 = -[p_1 L' + p_2 \Phi' + p_3 \Psi']/t' \tag{8d}$$

The last equation results because, for optimal path, $H = 0$ when H is independent of ϕ, which is free.

From the conditions for the maximum of H, we find that

$$\tan\theta = \left[-3B + \sqrt{9B^2 + 8A^2}\right] \Big/ 4A \tag{9}$$

where

$$A = 2p_1 + 2p_2 P + p_3 \Psi, \qquad B = p_3 P$$

Since θ is constrained to $[-\pi/2, \pi/2]$, Eq. (9) defines the control angle uniquely. Without loss of generality p_1 can be scaled to ± 1 initially, and the problem now is to find the solution of the system of Eqs. (4) and (8) with Eq. (9) and the boundary conditions (5) and (6). The selection of the appropriate initial values for p_2 and p_3 for the boundary-value problem is attempted by the CRS procedures,[6,7] the process of integration being terminated when the semi-major axis of the space vehicle attains that of the target planet.

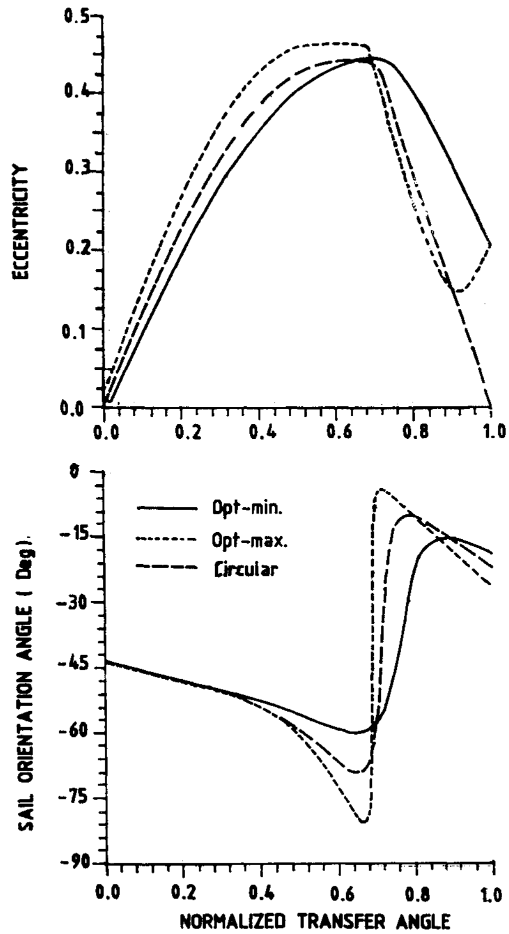

Fig. 1 Variations of eccentricity/sail-orientation angle.

Table 1 Optimum flight time for Earth-Mercury transfer

ν_1, deg	ϕ_F,[a] deg	T_F, days	ν_2, deg	Initial conditions[b] $-p_2$	$-p_3$
0	227.9	189	253.2	0.1217649481	0.6140174626
30	240.1	193	295.4	0.1021265557	0.5924984320
60	259.1	197	344.4	0.0827244157	0.5893887483
90	273.4	199	28.7	0.0685167447	0.5950214026
120	281.2	201	66.5	0.0602695883	0.6023809788
150	283.5	201	98.8	0.0596843427	0.6100085257
180	281.1	201	126.4	0.0676138049	0.6190052672
210	274.8	200	150.1	0.0825493763	0.6309420909
240	265.2	198	170.5	0.1003594772	0.6453469942
270	253.3	196	188.6	0.1159714343	0.6575774179
300	240.9	192	206.2	0.1261552650	0.6592233778
330	230.5	189	225.3	0.1294909722	0.6432071567

[a] Transfer angle corresponding to T_F.
[b] Truncated to 10 decimal places.
Earth: $a_1 = 1$ AU; $e_1 = 0.01675$, $\omega_1 = 101°.2$;
Mercury: $a_2 = 0.3871$ AU; $e_2 = 0.2056$, $\omega_2 = 75°.9$.

Optimization Algorithm

The CRS algorithm, an effective tool for global optimization, does not need computation of derivatives, but depends on the function F evaluations alone. It works even when the differentiability requirements cannot be ensured in the feasible domain V. To initiate this algorithm, no initial trial guess value, except for an estimate of V, is needed.

The CRS2 version of Price[6] as well as the one proposed by Prasad and Rao[7] works in two phases. In the first phase a set of N (the suggested value is $10n + 10$, where n is the number of variables involved) random feasible points are generated, and F is evaluated at each of these points. The information is stored as a matrix A of order $N \times (n+1)$. The maximum and minimum values F_M, F_R of F and the corresponding points M and R are then identified. In the second phase these random points are manipulated iteratively to yield a better candidate for the global solution. At each iteration, n arbitrary distinct points are chosen from A. A new point $T = 2G - R$, where G is the centroid of these points, is generated, and if T lies in V, then F_T is evaluated. If $F_T < F_M$, then F_M and M in A are replaced by F_T and T. Otherwise, T is discarded and a new T is again generated. Treating any replacement as a success and setting a minimum success rate of 0.5, the efficiency of the procedure is enhanced by making use of the secondary trial point $Q = (3G + R)/4$. If T or Q is a success, a third trial is also made with $X = 2.5$ (T or Q) $- 1.5R$, and the best (with the least F value) of T or Q or X is used for replacement in A. The iteration process is continued until the F_R value assumes the prefixed threshold value.

Illustrative Results

A computer code based on the foregoing methodology was developed in Fortran IV on a Control Data Corporation CYBER 170/730 computer. Optimum flight times and the profiles for θ for transfer from Earth to Mercury, Venus, and Martian orbits, in the ideal case when both the terminal orbits are regarded as circular and coplanar, are obtained and are found to be in good agreement with those reported in Refs. 1 and 2. For a typical illustration, details of the optimal Earth-Mercury transfer with the considerations of the ellipticity of both the terminal orbits are presented in Table 1 for $\alpha = 2$ mm/s^2. As may be inferred from Table 1, for any departure point on the Earth's orbit there is a corresponding true anomaly at arrival for rendezvous, and there are two extrema for T_F termed opt-min and opt-max occurring when v_1 is nearly 0 and 150 deg. The maximum variation in T_F is ~ 12 days. Graphical representations for e and θ are attempted in Fig. 1. However, it may be pointed out that, for smaller accelerations, rendezvous becomes difficult and even impossible for some regions of departure. For example, when $\alpha = 1$ mm/s^2, no rendezvous is found to be possible when v_1 is in the region of ~ 300–145 deg. Details relating to the convergence of the optimization algorithm are given in Ref. 8.

Conclusions

A convenient formulation for determining the effect of the eccentricities of the terminal orbits on the steering profile of a sail spacecraft for time-optimal rendezvous transfer between coplanar heliocentric orbits is presented. The optimal control strategy is attempted by conversion to a two-point boundary-value problem for a system of seven ordinary differential equations. Effectiveness of the CRS optimization technique for the solution is demonstrated.

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Significance of Modeling Internal Damping in the Control of Structures

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Introduction

THIS Note examines the significance and importance of modeling the internal damping of structures in designing feedback control laws for vibration suppression. The authors have in mind control problems for large flexible spacecraft. In the description of control of flexible space structure, the models used are large-order finite element models or partial differential equation models. However, the point of this Note can be made by examining simple one and two degree-of-freedom models as well as simple hybrid-distributed-parameter models. Damping in spacecraft is often ignored in control design because it is difficult to model and measure. Here several simple systems are examined to illustrate that control design with ignorance of damping has the potential for resulting in poor performance or even an unstable closed-loop response.

The negative effects of unmodeled damping are pronounced in systems that do not use collocated sensors and actuators. However, as shown in Ref. 1, better performance is obtained by noncollocated feedback laws. It has been shown that if the actuator dynamics are significant compared to those of the structure (often the case), then collocated sensors and actuators are not possible (see Ref. 1). Hence, a noncollocated feedback law is also considered here.

We also consider an example [the Rocket Propulsion Laboratory (RPL) structure] for which even the actuators (a tip jet nozzle and flexible hose) for a simple beam produce significant damping, which, if ignored in the basic system modeling, results in a model that cannot yield a reasonable time response using physically meaningful parameter values. Such a model also yields a less than satisfactory result when used in a control design.

Poorly Estimated Damping in Lumped-Parameter Systems

It is instructive to first consider simple velocity feedback control of a single degree-of-freedom oscillator. In this case, it is easy to verify that the estimate of the value of the open-loop damping coefficient greatly effects the closed-loop response. If the open-loop damping coefficient is underestimated, then the feedback control gain will be chosen to be smaller than actually needed and the desired closed-loop response will not be obtained. If, on the other hand, the open-loop damping coefficient is overestimated, a velocity feedback law with the objective of decreasing the system's speed of response (such as in motor control) can actually produce an unstable system (see Ref. 2).

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